

FIAN/TD-12/94

hep-th/9409190

Quantum Deformations of τ -functions, Bilinear Identities and Representation Theory¹

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ABSTRACT

This paper is a brief review of recent results on the concept of “generalized τ -function”, defined as a generating function of all the matrix elements in a given highest-weight representation of a universal enveloping algebra \mathcal{G} . Despite the differences from

¹Talk presented at the Workshop on Symmetries and Integrability of Difference Equations, Montreal, May, 1994

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the particular case of conventional τ -functions of integrable (KP and Toda lattice) hierarchies, these generic τ -functions also satisfy bilinear Hirota-like equations, which can be deduced from manipulations with intertwining operators. The main example considered in details is the case of quantum groups, when such τ -“functions” are not c -numbers but take their values in non-commutative algebras (of functions on the quantum group G). The paper contains only illustrative calculations for the simplest case of the algebra $SL(2)$ and its quantum counterpart $SL_q(2)$, as well as for the system of fundamental representations of $SL(n)$.

1 Introduction

The key object in the theory of classical integrable equations is the notion of τ -function. This function allows one to transform non-linear equations to bilinear homogeneous equations which are often called Hirota equations. Indeed, there is even more important property of the τ -function and Hirota equations – they can be easily extended to an infinite set of equations satisfied by the same (τ)-function of (infinitely many) variables (times). The most effective way to deal with this infinite set (and even to write down it) is to encapsulate it into few generating identities which we will name bilinear identities (BI). BI and τ -functions are the main content of the general approach to classical integrable systems as it has been developed in the papers of Kyoto school [1].

This paper, which is a review of the results obtained in collaboration with A.Gerasimov, S.Kharchev, S.Khoroshkin, D.Lebedev, A.Morozov and L.Vinet (see also [2, 3, 4, 5]), demonstrates how this approach can be generalized and reformulated in group theory terms so that the notion of τ -function which satisfies BI can be associated with an arbitrary group (and even quantum group) and with any highest-weight representation. Indeed, this generalized τ -function is just defined as a generating function of all matrix elements in a fixed representation. The standard KP (or Toda lattice) hierarchy τ -function [1] is associated with the group $GL(\infty)$ and its fundamental representations. Different KdV-type reductions are associated with the corresponding Kac-Moody algebras with unit central charge.

In the present paper, in order to illustrate some new specific features of the generalized τ -functions we consider the case of quantized algebras, mostly the simplest case of $SL_q(2)$ algebra. Another interesting example of extension of the standard theory – to Kac-Moody algebras with central charge greater than 1, or even to multi-loop algebras - is out of scope of the paper.

2 τ -function and bilinear identities

2.1 τ -function

Let us consider a universal enveloping algebra $U(\mathcal{G})$ and introduce a “ τ -function” for any Verma module V of this algebra as a generating function for all matrix elements $\langle \mathbf{k}|g|\mathbf{n}\rangle_V$:

$$\begin{aligned} \tau_V(t, \bar{t}|g) &\equiv \sum_{\substack{k_\alpha \geq 0 \\ n_\alpha \geq 0}} \prod_{\alpha > 0} \frac{t_\alpha^{k_\alpha}}{[k_\alpha]!} \frac{\bar{t}_\alpha^{n_\alpha}}{[n_\alpha]!} {}_V \langle \mathbf{k}_\alpha | g | \mathbf{n}_\alpha \rangle_V = \\ &= {}_V \langle \mathbf{0} | \prod_{\alpha > \mathbf{0}} e_q(t_\alpha T_\alpha) g \prod_{\alpha > \mathbf{0}} e_q(\bar{t}_\alpha T_{-\alpha}) | \mathbf{0} \rangle_V . \end{aligned} \tag{1}$$

Here $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$, $[n]! = [1][2] \dots [n]$, $e_q(x) = \sum_{n \geq 0} \frac{x^n}{[n]!}$. In the case of Lie algebras q -exponentials are substituted by the ordinary ones. $T_{\pm\alpha}$ are generators of positive/negative maximal nilpotent subalgebras $N(\mathcal{G})$ and $\bar{N}(\mathcal{G})$ of \mathcal{G} with suitably chosen ordering of positive roots α , and t_α , $\bar{t}_\alpha = t_{-\alpha}$ are the associated “time-variables”. Vacuum state is annihilated by all the positive generators: $T_\alpha |\mathbf{0}\rangle_V = 0$ for all $\alpha > 0$. Verma module $V =$

$\{|\mathbf{n}_\alpha\rangle_V = \prod_{\alpha>0} T_{-\alpha}^{n_\alpha} |\mathbf{0}\rangle_V\}$ is formed by the action of all the generators $T_{-\alpha}$ for all negative roots $-\alpha$ from maximal nilpotent subalgebra $\bar{N}(\mathcal{G})$.

Except for special circumstances all the $\alpha \in N(\mathcal{G})$ are involved, and since not all the $T_{-\alpha}$'s are commuting, the so defined τ -function has nothing to do with Hamiltonian integrability (see [6] for detailed description of the specifics of $k = 1$ Kac-Moody algebras in this context). However, this appears to be the only property of conventional τ -functions which is not preserved by our general definition. The "non-Cartanian" τ -function and "non-Cartanian" hierarchy still reduce to the standard integrable hierarchy when restricting onto the fundamental representations which can be completely generated by the commutative subalgebra of $N(\mathcal{G})$.

Example 1. $GL(\infty)$, V = fundamental representation

Let us consider $GL(\infty)$ with Dynkin diagram infinite in both directions. Each vertex on the diagram corresponds to a fundamental representation $F^{(n)}$ with arbitrary fixed origin $n = 0$ (n can be both positive and negative). This example admits free fermionic formulation [1] (see also [7, 6, 8]):

$$\tau_n(t, \bar{t}|g) = \langle F^{(n)} | e^{H\{t\}} g e^{\bar{H}\{\bar{t}\}} | F^{(n)} \rangle \quad (2)$$

with

$$H\{t\} = \sum_{n>0} t_n J_{+n}, \quad \bar{H}\{\bar{t}\} = \sum_{n>0} \bar{t}_n J_{-n}, \quad J_n = \sum_{k=-\infty}^{\infty} \psi_k \psi_{k+n}^*, \quad g = \exp \left(\sum_{m,n} \mathcal{A}_{mn} \psi_n^* \psi_m \right),$$

$$\{\psi_i, \psi_j^*\} = \delta_{ij}, \quad \{\psi_i, \psi_j\} = \{\psi_i^*, \psi_j^*\} = 0; \quad \psi_k^* | F^{(n)} \rangle = 0, \quad \text{for } k \geq n, \quad \psi_k | F^{(n)} \rangle = 0 \quad \text{for } k < n, \quad (3)$$

where two sets of currents J_n with positive and negative n give a manifest realization of the commutative nilpotent subalgebras $N(\mathcal{G})$ and $\bar{N}(\mathcal{G})$ accord-

ingly, and arbitrary generators of algebra can be realized as bilinears $\psi_k \psi_{k+n}^*$. In this example fermions intertwine different fundamental representations.

Example 2. $SL(2)$

In this case, the calculation is very simple for arbitrary highest weight representation with spin λ :

$$\tau_\lambda = {}_\lambda \langle 0 | e^{tT_-} g e^{\bar{t}T_+} | 0 \rangle_\lambda = (a + b\bar{t} + ct + dt\bar{t})^{2\lambda}, \quad (4)$$

where the group element g is parameterized by three parameters:

$$g = e^{x_+ T_+} e^{x_0 T_0} e^{x_- T_-}, \quad (5)$$

and

$$a \equiv e^{\frac{1}{2}x_0} + x_+ x_- e^{-\frac{1}{2}x_0}, \quad b \equiv x_+ e^{-\frac{1}{2}x_0}, \quad c \equiv e^{-\frac{1}{2}x_0} x_-, \quad d \equiv e^{-\frac{1}{2}x_0}, \quad (6)$$

i.e.

$$ad - bc = 1. \quad (7)$$

2.2 Vertex operators and bilinear identities

Vertex operators

1. The starting point is embedding of Verma module \hat{V} into the tensor product $V \otimes W$, where W is some irreducible finite-dimensional representation of \mathcal{G} .³ Once V and W are specified, there is only finite number of choices for \hat{V} .

³In the case of Affine algebra, one should use evaluation representation – zero charge representation induced from finite-dimensional one – see the definition of vertex operator in [9, 10].

Now we define right vertex operator of the W -type as homomorphism of \mathcal{G} -modules:

$$E_R : \hat{V} \longrightarrow V \otimes W. \quad (8)$$

This intertwining operator can be explicitly continued to the whole representation once this is constructed for its vacuum (highest-weight) state:

$$\hat{V} = \left\{ |\mathbf{n}_\alpha\rangle_{\hat{V}} = \prod_{\alpha>0} (\Delta(T_{-\alpha})^{n_\alpha} |\mathbf{0}\rangle_{\hat{V}}) \right\}, \quad (9)$$

where comultiplication Δ provides the action of \mathcal{G} on the tensor product of representations, and

$$|\mathbf{0}\rangle_{\hat{V}} = \left(\sum_{\{p_\alpha, i_\alpha\}} A\{p_\alpha, i_\alpha\} \left(\prod_{\alpha>0} (T_{-\alpha})^{p_\alpha} \otimes (T_{-\alpha})^{i_\alpha} \right) \right) |\mathbf{0}\rangle_V \otimes |\mathbf{0}\rangle_W. \quad (10)$$

For finite-dimensional W 's, this provides every $|\mathbf{n}_\alpha\rangle_{\hat{V}}$ in a form of *finite* sums of states $|\mathbf{m}_\alpha\rangle_V$ with coefficients, taking values in elements of W .

2. The next step is to take another triple, defining left vertex operator,

$$E'_L : \hat{V}' \longrightarrow W' \otimes V', \quad (11)$$

such that the product $W \otimes W'$ contains *unit* representation of \mathcal{G} .

Bilinear identities in terms of universal enveloping algebra

The derivation of BI consists of two steps. The first one is to consider the projection to this unit representation

$$\pi : W \otimes W' \longrightarrow I \quad (12)$$

explicitly provided by multiplication of any element of $W \otimes W'$ by

$$\pi = {}_W\langle \mathbf{0} | \otimes {}_{W'}\langle \mathbf{0} | \left(\sum_{\{i_\alpha, i'_\alpha\}} \pi\{i_\alpha, i'_\alpha\} \left(\prod_{\alpha>0} (T_{+\alpha})^{i_\alpha} \otimes (T_{+\alpha})^{i'_\alpha} \right) \right) \quad (13)$$

Using this projection, one can build a new intertwining operator

$$\Gamma : \hat{V} \otimes \hat{V}' \xrightarrow{E_R \otimes E'_L} V \otimes W \otimes W' \otimes V' \xrightarrow{I \otimes \pi \otimes I} V \otimes V', \quad (14)$$

which possesses the property

$$\Gamma(g \otimes g) = (g \otimes g)\Gamma \quad (15)$$

for any group element g such that

$$\Delta(g) = g \otimes g. \quad (16)$$

Put it differently, the space $W \otimes W'$ contains a canonical element of pairing $w_i \otimes w^i$ which commutes with the action of $\Delta(g)$. This means that the operator $\sum_i E_i \otimes E^i : V \otimes V' \longrightarrow \hat{V} \otimes \hat{V}'$ ($E_i \equiv E(w_i)$, $E^i \equiv E(w^i)$) commutes with $\Delta(g)$.

Identity (15) is nothing but an algebraic form of BI. To transform this to the differential (difference) form (the second step of the derivation), one needs to use the second line of definition (1) and to average identity (15) with the evolution exponentials over the universal enveloping algebra. Then one gets a BI for the averages like (1) (there are many equivalent identities, in accordance with many possible choices of the states which one averages over), but with additional insertions of E_i 's and E^i 's. Using the commutation relations of these intertwiners with the generators of the algebra, one can push E_i 's out to the proper vacuums. This procedure of pushing can be imitated by the action of some differential (difference) operators, and, as a result, one gets, instead of averaged eq.(15), differential (difference) BI.⁴

⁴In fact, in order to represent the result of pushing by a differential or difference operator

This second step of the derivation is very group-dependent, and we illustrate it below in some concrete examples.

Example 1. $GL(\infty)$, V =fundamental representation

We will show in section 5.1 that the intertwining operators between fundamental representations in this example are fermions (see example 1 in the previous subsection). The operator $\Gamma = \sum_i \psi_i \psi^i = \sum_i \psi_i \psi_i^*$. Then, BI (15) gets the form

$$\begin{aligned} & \sum_i \langle F^{(n+1)} | e^{H\{t\}} \psi_i g e^{\bar{H}\{\bar{t}\}} | F^{(n)} \rangle \cdot \langle F^{(m-1)} | e^{H\{t'\}} \psi_i^* g e^{\bar{H}\{\bar{t}'\}} | F^{(m)} \rangle = \\ & = \sum_i \langle F^{(n+1)} | e^{H\{t\}} g \psi_i e^{\bar{H}\{\bar{t}\}} | F^{(n)} \rangle \cdot \langle F^{(m-1)} | e^{H\{t'\}} g \psi_i^* e^{\bar{H}\{\bar{t}'\}} | F^{(m)} \rangle, \end{aligned} \quad (17)$$

where one averages (15) over the states $\langle F^{(n+1)} | e^{H(t)} \otimes \langle F^{(m-1)} | e^{H(t')}$ and $e^{\bar{H}(\bar{t})} | F^{(n)} \rangle \otimes e^{\bar{H}(\bar{t}')} | F^{(m)} \rangle$. One can rewrite (17) through the free fermion fields $\psi(z) \equiv \sum_i \psi_i z^i$ and $\psi(z)^* \equiv \sum_i \psi_i^* z^{-i}$:

$$\begin{aligned} & \oint_{\infty} \frac{dz}{z} \langle F^{(n+1)} | e^{H\{t\}} \psi(z) g e^{\bar{H}\{\bar{t}\}} | F^{(n)} \rangle \cdot \langle F^{(m-1)} | e^{H\{t'\}} \psi^*(z) g e^{\bar{H}\{\bar{t}'\}} | F^{(m)} \rangle = \\ & = \oint_0 \frac{dz}{z} \langle F^{(n+1)} | e^{H\{t\}} g \psi(z) e^{\bar{H}\{\bar{t}\}} | F^{(n)} \rangle \cdot \langle F^{(m-1)} | e^{H\{t'\}} g \psi^*(z) e^{\bar{H}\{\bar{t}'\}} | F^{(m)} \rangle. \end{aligned} \quad (18)$$

Now, using the relations

$$\begin{aligned} \langle F^{(n)} | e^{H(t)} \psi(z) &= z^{n-1} \langle F^{(n-1)} | \exp[H(t_k - \frac{1}{kz^k})] \equiv z^{n-1} \hat{X}(z, t) \langle F^{(n-1)} | e^{H(t)}, \\ \langle F^{(n)} | e^{H(t)} \psi^*(z) &= z^{-n} \langle F^{(n+1)} | \exp[H(t_k + \frac{1}{kz^k})] \equiv z^{-n} \hat{X}^*(z, t) \langle F^{(n+1)} | e^{H(t)}, \end{aligned} \quad (19)$$

one needs to choose properly the generating coefficients in definition (1). The choice accepted in the paper is not unique.

where

$$\hat{X}(z, t) = e^{\xi(z, t)} e^{-\xi(z^{-1}, \tilde{\partial}_t)}, \quad \hat{X}^*(z, t) = e^{-\xi(z, t)} e^{\xi(z^{-1}, \tilde{\partial}_t)}, \quad \xi(z, t) \equiv \sum_i z^i t_i, \quad \tilde{\partial}_{t_k} \equiv \frac{1}{k} \partial_{t_k} \quad (20)$$

(and similarly for the right vacuum state), one finally gets the integral form of BI:

$$\oint_{\infty} dz z^{n-m} \hat{X}(z, t) \tau_n(t, \bar{t}) \hat{X}^*(z, t') \tau_m(t', \bar{t}') = \oint_0 \frac{dz}{z^2} z^{n-m} \hat{X}(z^{-1}, \bar{t}) \tau_{n+1}(t, \bar{t}) \hat{X}^*(z^{-1}, \bar{t}') \tau_{m-1}(t', \bar{t}'), \quad (21)$$

which can be easily transformed to an infinite set of differential equations by expanding to the degrees of time differences $t_i - t'_i$ etc. [1].

Bilinear identities in terms of algebra of functions

Instead of averaging of (15) over the universal enveloping algebra when deriving BI, one can work in terms of matrix elements, i.e. at the dual language of the algebra of functions. Indeed, let us take a matrix element of (15) between four states,

$${}_{V'} \langle k' | {}_V \langle k | (g \otimes g) \Gamma | n \rangle_{\hat{V}} | n' \rangle_{\hat{V}'} = {}_{V'} \langle k' | {}_V \langle k | \Gamma (g \otimes g) | n \rangle_{\hat{V}} | n' \rangle_{\hat{V}'} . \quad (22)$$

The action of operator Γ can be represented as

$$\Gamma | n \rangle_{\hat{\lambda}} | n' \rangle_{\hat{\lambda}'} = \sum_{l, l'} | l \rangle_{\lambda} | l' \rangle_{\lambda'} \Gamma(l, l' | n, n'), \quad (23)$$

and (22) turns into

$$\sum_{m, m'} \Gamma(k, k' | m, m') \frac{||k||_{\hat{\lambda}}^2 ||k'||_{\hat{\lambda}'}^2}{||m||_{\hat{\lambda}}^2 ||m'||_{\hat{\lambda}'}^2} \langle m | g | n \rangle_{\hat{\lambda}} \langle m' | g | n' \rangle_{\hat{\lambda}'} = \sum_{l, l'} \langle k | g | l \rangle_{\lambda} \langle k' | g | l' \rangle_{\lambda'} \Gamma(l, l' | n, n'). \quad (24)$$

In order to rewrite this as a difference equation, we use the first line in definition (1) of τ -function. Then, one can write down the generating formula for equation (24), using the manifest form (30)-(31) of matrix elements $\Gamma(l, l'|n, n')$. We illustrate this approach in the simplest example of $SL_q(2)$ in section 3.1.

3 $SL_q(2)$ case

3.1 Bilinear identities

As an example of the technique developed in the previous section, we discuss here BI and their solutions in the case of the simplest quantum group $SL_q(2)$.

The notations: algebra $U_q(SL(2))$ has generators T_+ , T_- and T_0 with commutation relations

$$q^{T_0}T_{\pm}q^{-T_0} = q^{\pm 1}T_{\pm}, \quad [T_+, T_-] = \frac{q^{2T_0} - q^{-2T_0}}{q - q^{-1}}, \quad (25)$$

and comultiplication

$$\Delta(T_{\pm}) = q^{T_0} \otimes T_{\pm} + T_{\pm} \otimes q^{-T_0}, \quad \Delta(q^{T_0}) = q^{T_0} \otimes q^{T_0}. \quad (26)$$

Verma module V_{λ} with highest weight λ (not obligatory half-integer), consists of the elements

$$|n\rangle_{\lambda} \equiv T_-^n |0\rangle_{\lambda}, \quad n \geq 0, \quad (27)$$

such that

$$\begin{aligned}
T_-|n\rangle_\lambda &= |n+1\rangle_\lambda, \quad T_0|n\rangle_\lambda = (\lambda - n)|n\rangle_\lambda, \quad T_+|n\rangle_\lambda \equiv b_n(\lambda)|n-1\rangle_\lambda, \\
b_n(\lambda) &= [n][2\lambda + 1 - n], \quad [x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}, \quad ||n||_\lambda^2 \equiv {}_\lambda\langle n|n\rangle_\lambda = \frac{[n]! \Gamma_q(2\lambda + 1)}{\Gamma_q(2\lambda + 1 - n)} \stackrel{\lambda \in \mathbf{Z}/2}{=} \frac{[2\lambda]![n]!}{[2\lambda - n]!}.
\end{aligned} \tag{28}$$

Now one could use the manifest commutation relations of intertwiners with generators of $U_q(SL(2))$ to obtain BI along the line of example 1, but instead, for the illustration of another approach, we manifestly calculate matrix elements of the operator Γ .

Let us take for W an irreducible spin- $\frac{1}{2}$ representation of $U_q(SL(2))$. Then $\hat{V} = V_{\lambda \pm \frac{1}{2}}$, $V = V_\lambda$.

In order to obtain matrix elements of Γ , one should project the tensor product of two different W 's onto singlet state $S = |+\rangle|-\rangle - q|-\rangle|+\rangle$:

$$(A|+\rangle + B|-\rangle) \otimes (|+\rangle C + |-\rangle D) \longrightarrow AD - qBC. \tag{29}$$

With our choice of W we can now consider two different cases:

(A) both $\hat{V} = V_{\lambda - \frac{1}{2}}$ and $\hat{V}' = V_{\lambda' - \frac{1}{2}}$, or

(B) $\hat{V} = V_{\lambda - \frac{1}{2}}$ and $\hat{V}' = V_{\lambda' + \frac{1}{2}}$.

The result of calculation gives the following matrix elements of the projection⁵:

⁵Hereafter we omit the symbol of tensor product from the notations of the states $|+\rangle \otimes |0\rangle_\lambda$ etc.

Case A:

$$|n\rangle_{\lambda-\frac{1}{2}}|n'\rangle_{\lambda'-\frac{1}{2}} \longrightarrow q^{\frac{n'-n-1}{2}} \left([n' - 2\lambda']q^{\lambda'}|n+1\rangle_{\lambda}|n'\rangle_{\lambda'} - [n - 2\lambda]q^{-\lambda}|n\rangle_{\lambda}|n'+1\rangle_{\lambda'} \right). \quad (30)$$

Case B:

$$|n\rangle_{\lambda+\frac{1}{2}}|n'\rangle_{\lambda'-\frac{1}{2}} \longrightarrow q^{\frac{n'-n-1}{2}} \left([n' - 2\lambda']q^{\lambda'}|n\rangle_{\lambda}|n'\rangle_{\lambda'} - [n]q^{+\lambda+1}|n-1\rangle_{\lambda}|n'+1\rangle_{\lambda'} \right). \quad (31)$$

Then, one can write down the generating formula for equation (24), using the manifest form (30)-(31) of matrix elements $\Gamma(l, l'|n, n')$:

Case A:

$$\begin{aligned} & \sqrt{M_t^- M_{t'}^+} \left(q^{\lambda'} D_t^{(0)} \bar{t}' D_{\bar{t}'}^{(2\lambda')} - q^{-\lambda} \bar{t} D_t^{(2\lambda)} D_{\bar{t}'}^{(0)} \right) \tau_{\lambda}(t, \bar{t}|g) \tau_{\lambda'}(t', \bar{t}'|g) = \\ & = [2\lambda][2\lambda'] \sqrt{M_t^- M_{t'}^+} \left(q^{-(\lambda+\frac{1}{2})} t' - q^{(\lambda'+\frac{1}{2})} t \right) \tau_{\lambda-\frac{1}{2}}(t, \bar{t}|g) \tau_{\lambda'-\frac{1}{2}}(t', \bar{t}'|g). \end{aligned} \quad (32)$$

Here $D_t^{(\alpha)} \equiv \frac{q^{-\alpha} M_t^+ - q^{\alpha} M_t^-}{(q - q^{-1})t}$ and M^{\pm} are multiplicative shift operators, $M_t^{\pm} f(t) = f(q^{\pm 1}t)$.

Case B:

$$\begin{aligned} & \sqrt{M_t^- M_{t'}^+} \left(q^{\lambda'} \bar{t}' D_{\bar{t}'}^{(2\lambda')} - q^{(\lambda+1)} \bar{t} D_t^{(0)} \right) \tau_{\lambda}(t, \bar{t}|g) \tau_{\lambda'}(t', \bar{t}'|g) = \\ & = \frac{[2\lambda']}{[2\lambda+1]} \sqrt{M_t^- M_{t'}^+} \left(q^{\lambda'} t D_t^{(2\lambda+1)} - q^{\lambda} t' D_t^{(0)} \right) \tau_{\lambda+\frac{1}{2}}(t, \bar{t}|g) \tau_{\lambda'-\frac{1}{2}}(t', \bar{t}'|g). \end{aligned} \quad (33)$$

The classical limits of these equations are

Case A:

$$\left(2\lambda \frac{\partial}{\partial \bar{t}'} - 2\lambda' \frac{\partial}{\partial \bar{t}} + (\bar{t}' - \bar{t}) \frac{\partial^2}{\partial \bar{t} \partial \bar{t}'} \right) \tau_{\lambda}(t, \bar{t}|g) \tau_{\lambda'}(t', \bar{t}'|g) = 4\lambda\lambda' (t' - t) \tau_{\lambda-\frac{1}{2}}(t, \bar{t}|g) \tau_{\lambda'-\frac{1}{2}}(t', \bar{t}'|g). \quad (34)$$

Case B:

$$\left[(\bar{t}' - \bar{t}) \frac{\partial}{\partial \bar{t}'} - 2\lambda' \right] \tau_\lambda(t, \bar{t}|g) \tau_{\lambda'}(t', \bar{t}'|g) = \frac{2\lambda'}{2\lambda + 1} \left[(t - t') \frac{\partial}{\partial t} - 2\lambda - 1 \right] \tau_{\lambda+\frac{1}{2}}(t, \bar{t}|g) \tau_{\lambda'-\frac{1}{2}}(t', \bar{t}'|g). \quad (35)$$

3.2 Solutions

Classical limit and Liouville equation

We begin with considering the solutions to the classical BI. We only look at eq.(34), as any solution to this equation satisfies all other BI, say, eq.(35) or that obtained within a different choice of V , \hat{V} and W . The general solution is 3-parametric one and certainly coincides with the result (4) of the direct calculation of section 2.1.

This solution (4) at $\lambda = \frac{1}{2}$ (fundamental representation) seems to have a little to do with the solution to the Liouville equation which is also sometimes associated with $SL(2)$. However, there is a connection: though the Liouville equation has much more ample set of solutions, eq.(34) is contained in this set. Let us see how additional limitations arise within our approach.

First of all, like all Hirota type equations, (34) can be rewritten as a (system of) ordinary differential equations, when expanded in powers of $\epsilon = \frac{1}{2}(t - t')$ and $\bar{\epsilon} = \frac{1}{2}(\bar{t} - \bar{t}')$. For example, for $\lambda = \lambda'$ we obtain from (34):

$$\begin{aligned} \text{coefficient in front of } \epsilon : \quad & \partial \tau_\lambda \bar{\partial} \tau_\lambda - \tau_\lambda \partial \bar{\partial} \tau_\lambda = 2\lambda \tau_{\lambda-\frac{1}{2}}^2; \\ \text{coefficient in front of } \bar{\epsilon} : \quad & 2\lambda \tau_\lambda \bar{\partial}^2 \tau_\lambda = (2\lambda - 1)(\bar{\partial} \tau_\lambda)^2; \end{aligned} \quad (36)$$

...

If $\lambda = \frac{1}{2}$, the first one of these is just the Liouville equation:

$$\partial\tau_{\frac{1}{2}}\bar{\partial}\tau_{\frac{1}{2}} - \tau_{\frac{1}{2}}\partial\bar{\partial}\tau_{\frac{1}{2}} = \tau_0^2 = 1, \quad (37)$$

or

$$\partial\bar{\partial}\phi = 2e^\phi, \quad \tau_{\frac{1}{2}} = e^{-\phi/2}, \quad (38)$$

while the second one,

$$\bar{\partial}^2\tau_{\frac{1}{2}} = 0, \quad (39)$$

is a very restrictive constraint. Its role is to reduce the huge set of solutions to the Liouville equation,

$$\tau_{\frac{1}{2}}(t, \bar{t}|g) = (1 + A(t)B(\bar{t})) \left[\frac{\partial A}{\partial t} \frac{\partial B}{\partial \bar{t}} \right]^{-\frac{1}{2}}, \quad (40)$$

parametrized by two *arbitrary* functions $A(t)$ and $B(\bar{t})$, to the 3-parametric family (4). In the language of infinite-dimensional Grassmannian there are infinitely many ways to embed $SL(2)$ group into $GL(\infty)$ – and all the cases correspond to solutions (40) (with some $A(t)$ and $B(\bar{t})$) to the $SL(2)$ reduced Toda-lattice hierarchy (i.e. Liouville equation), - but constraint (39) specifies very concrete embedding: that in the left upper corner of $GL(\infty)$ matrix and is associated with *linear* functions $A(t)$ and $B(\bar{t})$.

In terms of example 1 the general Liouville solution corresponds to the matrix \mathcal{A}_{mn} from (3) of the form $\sum_{i=1,2} F_n^{(i)} G_m^{(i)}$ with arbitrary coefficients $F_n^{(i)}$ and $G_m^{(i)}$ (i.e. to a matrix of the rank 2). Then, the corresponding

element g rotates fermions in two-dimensional invariant subspace:

$$\begin{aligned}
g\psi_i g^{-1} &= \left(\int dt \frac{e^{-tx}}{\sqrt{\partial A}} \int dx x^{i-1} \right) \cdot \sum_k \left(\int d\bar{t} \frac{e^{-\bar{t}y}}{\sqrt{\partial B}} \int dy y^{k-1} \right) \psi_k + \\
&+ \left(\int dt \frac{e^{-tx} A}{\sqrt{\partial A}} \int dx x^{i-1} \right) \cdot \sum_k \left(\int d\bar{t} \frac{e^{-\bar{t}y} B}{\sqrt{\partial B}} \int dy y^{k-1} \right) \psi_k \equiv f_i \Psi^{(1)} + g_i \Psi^{(2)}
\end{aligned} \tag{41}$$

in contrast to the general law [1]

$$g\psi_i g^{-1} = \sum_j R_{ij} \psi_j. \tag{42}$$

Here the linear combinations of fermions $\Psi_i^{(1,2)}$ depend on concrete choice of g , i.e. on the functions A and B . The whole variety of Liouville solutions is given by different choices of the coefficients $F_n^{(i)}$, $G_m^{(i)}$ etc. (they are connected with the moments of Fourier components of the functions $A(t)$ and $B(\bar{t})$ like (41)), and different choices are related by (outer) $GL(\infty)$ automorphisms of the $SL(2)$ system (see (41)). Choosing only 2 first non-zero coefficients, one returns to the case of the present paper.

Quantum commutative τ -function

Now let us look at the solution to the quantum equation (32). One can easily check that

$$\tau_\lambda = \left[\alpha + \frac{1}{\alpha} t\bar{t} \right]^{2\lambda} \equiv \sum_{i \geq 0} \frac{\Gamma_q(2\lambda + 1)}{\Gamma_q(2\lambda + 1 - i)} \frac{\alpha^{2\lambda - 2i} (t\bar{t})^i}{[i]!} \tag{43}$$

does indeed satisfy (32), since

$$\begin{aligned}
D_t^{(0)} \left[\alpha + \frac{1}{\alpha} t\bar{t} \right]^{2\lambda} &= \frac{1}{\alpha} [2\lambda] \left[\alpha + \frac{1}{\alpha} t\bar{t} \right]^{2\lambda-1} \bar{t}, \\
t D_t^{(2\lambda)} \left[\alpha + \frac{1}{\alpha} t\bar{t} \right]^{2\lambda} &= -\alpha [2\lambda] \left[\alpha + \frac{1}{\alpha} t\bar{t} \right]^{2\lambda-1}.
\end{aligned} \tag{44}$$

However, this is only 1-parametric solution, in contrast to the classical case. This is due to the fact that, of all elements of $U_q(SL(2))$, the only Cartan element has the proper comultiplication law (16), while in the classical case there is the 3-parametric family of such elements (5).

Quantum non-commutative τ -function

The way to construct whole family of solutions to the quantum BI is to consider the non-commutative τ -function. Indeed, the first line of definition (1) implies that the τ -function takes its values in the algebra of functions on $SL_q(2)$, i.e. is non-commutative quantity. For example, in fundamental representation it is equal to

$$\tau_{\frac{1}{2}} = \langle +|g|+ \rangle + \bar{t}\langle +|g|- \rangle + t\langle -|g|+ \rangle + t\bar{t}\langle -|g|- \rangle = a + b\bar{t} + ct + dt\bar{t}, \quad (45)$$

where generators a, b, c, d of the algebra of functions $A(SL_q(2))$ are the entries of the matrix

$$\mathcal{T} = \begin{pmatrix} ab \\ cd \end{pmatrix}, \quad ad - qbc = 1 \quad (46)$$

with the commutation relations dictated by $\mathcal{T}\mathcal{T}\mathcal{R} = \mathcal{R}\mathcal{T}\mathcal{T}$ equation [11]

$$ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb, \quad ad - da = (q - q^{-1})bc. \quad (47)$$

In order to obtain this non-commutative τ -function from the second line of (1) and, simultaneously, to enlarge the number of group elements satisfying condition (16), one needs to consider g as an element of the universal enveloping algebra given over *non-commutative* ring instead of complex numbers. This ring is just $A_q(SL(2))$ (see the next section).

In order to construct the non-commutative τ -function for any representation with spin λ , one can expand this representation to the representations with spins $\lambda - \frac{1}{2}$ and $\frac{1}{2}$:

$$\begin{aligned} {}_{\lambda}\langle k|g|n\rangle_{\lambda} &= q^{-\frac{k+n}{2}} \left[{}_{\lambda-\frac{1}{2}}\langle k|g|n\rangle_{\lambda-\frac{1}{2}} \langle +|g|+\rangle + q^{\lambda}[n] {}_{\lambda-\frac{1}{2}}\langle k|g|n-1\rangle_{\lambda-\frac{1}{2}} \langle +|g|-\rangle + \right. \\ &\quad \left. + q^{\lambda}[k] {}_{\lambda-\frac{1}{2}}\langle k-1|g|n\rangle_{\lambda-\frac{1}{2}} \langle -|g|+\rangle + q^{2\lambda}[k][n] {}_{\lambda-\frac{1}{2}}\langle k-1|g|n-1\rangle_{\lambda-\frac{1}{2}} \langle -|g|-\rangle \right]. \end{aligned} \quad (48)$$

Applying this procedure recursively, one gets:

$$\begin{aligned} \tau_{\lambda}(t, \bar{t}|g) &= \tau_{\lambda-\frac{1}{2}}(q^{-\frac{1}{2}}t, q^{-\frac{1}{2}}\bar{t}|g) \tau_{\frac{1}{2}}(q^{\lambda-\frac{1}{2}}t, q^{\lambda-\frac{1}{2}}\bar{t}|g) = \\ &\stackrel{\text{if } \lambda \in \mathbf{Z}/2}{=} \tau_{\frac{1}{2}}(q^{\frac{1}{2}-\lambda}t, q^{\frac{1}{2}-\lambda}\bar{t}|g) \tau_{\frac{1}{2}}(q^{\frac{3}{2}-\lambda}t, q^{\frac{3}{2}-\lambda}\bar{t}|g) \dots \tau_{\frac{1}{2}}(q^{\lambda-\frac{1}{2}}t, q^{\lambda-\frac{1}{2}}\bar{t}|g). \end{aligned} \quad (49)$$

4 Universal T-operator

4.1 Universal T-operator

Let us manifestly describe the construction of "the group element" over the non-commutative ring. More precisely, we describe such an element $g \in U_q(G) \otimes A(G)$ of the tensor product of the universal enveloping algebra $U_q(G)$ and its dual $A(G)$ that

$$\Delta_U(g) = g \otimes_U g \in A(G) \otimes U_q(G) \otimes U_q(G). \quad (50)$$

In order to construct this element [11, 12, 5, 13], we fix some basis $T^{(\alpha)}$ in $U_q(G)$. Between $U_q(G)$ and $A(G)$ there is a non-generated pairing $\langle \dots \rangle$. We fix the basis $X^{(\beta)}$ in $A(G)$ orthonormal to $T^{(\alpha)}$ with respect to this pairing.

Then, the sum

$$\mathbf{T} \equiv \sum_{\alpha} X^{(\alpha)} \otimes T^{(\alpha)} \in A(G) \otimes U_q(G) \quad (51)$$

is just the group element we need. It is called universal \mathbf{T} -matrix (as it is intertwined by the universal \mathcal{R} -matrix).

In order to prove (50), one should remark that the matrices $M_{\gamma}^{\alpha\beta}$ and $D_{\beta\gamma}^{\alpha}$ giving rise to multiplication and comultiplication laws in $U_q(G)$ respectively

$$T^{(\alpha)} \cdot T^{(\beta)} \equiv M_{\gamma}^{\alpha\beta} T^{(\gamma)}, \quad \Delta(T^{(\alpha)}) \equiv D_{\beta\gamma}^{\alpha} T^{(\beta)} \otimes T^{(\gamma)} \quad (52)$$

induce, inversly, the comultiplication and multiplication in the dual algebra $A(G)$:

$$\begin{aligned} D_{\beta\gamma}^{\alpha} &= \langle \Delta(T^{(\alpha)}), X^{(\beta)} \otimes X^{(\gamma)} \rangle \equiv \langle T^{(\alpha)}, X^{(\beta)} \cdot X^{(\gamma)} \rangle, \\ M_{\gamma}^{\alpha\beta} &= \langle T^{(\alpha)} T^{(\beta)}, X^{(\gamma)} \rangle = \langle T^{(\alpha)} \otimes T^{(\beta)}, \Delta(X^{(\gamma)}) \rangle. \end{aligned} \quad (53)$$

Then,

$$\Delta_U(\mathbf{T}) = \sum_{\alpha} X^{(\alpha)} \otimes \Delta_U(T^{(\alpha)}) = \sum_{\alpha, \beta, \gamma} D_{\beta\gamma}^{\alpha} X^{(\alpha)} \otimes T^{(\beta)} \otimes T^{(\gamma)} = \sum_{\beta, \gamma} X^{(\beta)} X^{(\gamma)} \otimes T^{(\beta)} \otimes T^{(\gamma)} = \mathbf{T} \otimes_U \mathbf{T} \quad (54)$$

This is the first defining property of the universal \mathbf{T} -operator, which coincides with the classical one. The second property, which allows one to consider \mathbf{T} as generating the "true" group, is the group composition law $g \cdot g' = g''$ given by the map:

$$g \cdot g' \equiv \mathbf{T} \otimes_A \mathbf{T} \in A(G) \otimes A(G) \otimes U_q(G) \longrightarrow g'' \in A(G) \otimes U_q(G). \quad (55)$$

This map is canonically given by the comultiplication and is again the uni-

versal **T**-operator:

$$\mathbf{T} \otimes_A \mathbf{T} = \sum_{\alpha, \beta} X^{(\alpha)} \otimes X^{(\beta)} \otimes T^{(\alpha)} T^{(\beta)} = \sum_{\alpha, \beta, \gamma} M_{\alpha, \beta}^{\gamma} X^{(\alpha)} \otimes X^{(\beta)} \otimes T^{(\gamma)} = \sum_{\alpha} \Delta(X^{(\alpha)}) \otimes T^{(\alpha)}, \quad (56)$$

i.e.

$$g \equiv \mathbf{T}(X, T), \quad g' \equiv \mathbf{T}(X', T), \quad g'' \equiv \mathbf{T}(X'', T),$$

$$X \equiv \{X^{(\alpha)} \otimes I\} \in A(G) \otimes I, \quad X' \equiv \{I \otimes X^{(\alpha)}\} \in I \otimes A(G), \quad X'' \equiv \{\Delta(X^{(\alpha)})\} \in A(G) \otimes A(G) \quad (57)$$

4.2 Manifest construction of **T**-operator for $SL_q(2)$

In order to get more compact formulas let us redefine the generators of $U_q(SL(2))$ to obtain non-symmetric comultiplication law:

$$T_+ \longrightarrow T_+ q^{-T_0}, \quad T_- \longrightarrow q^{T_0} T_-, \quad \Delta(T_+) = I \otimes T_+ + T_+ \otimes q^{-2T_0}, \quad \Delta(T_-) = T_- \otimes I + q^{2T_0} \otimes T_-. \quad (58)$$

From now on, we also change the definitions of q -numbers $[n]_q \equiv \frac{1-q^n}{1-q}$ and, respectively, q -exponentials.

Now fix the basis $T^{(\alpha)} = T_+^i T_0^j T_-^k$ in $U_q(SL(2))$. Then, from the coproduct of $T^{(\alpha)}$ one can calculate matrix $D_{\beta\gamma}^{\alpha}$ (53) and manifestly calculate the orthonormal basis of $X^{(\alpha)}$:

$$X^{(\alpha)} = \frac{x_+^i}{[i]_{q^{-1}}!} \frac{x_0^j}{j!} \frac{x_-^k}{[k]_q!}, \quad (59)$$

where the generating elements x_{\pm} , x_0 satisfy the Borel Lie algebra

$$[x_0, x_{\pm}] = (\ln q) x_{\pm}, \quad [x_+, x_-] = 0. \quad (60)$$

Thus,

$$\mathbf{T} = e_{q^{-1}}^{x_+ T_+} e^{x_0 T_0} e_q^{x_- T_-}. \quad (61)$$

This expression, indeed, very resembles element of the classical Lie group and hints that the generators T_α in the definition of the τ -function (1) should be rather taken as elements of the extended algebra $U(G) \times A(G)$. Then, time variables are nothing but parameters of c -number automorphisms of the dual algebra $A(G)$. Say, in example (60) x_+ and x_- can be multiplied by c -number factors, and the third c -number parameter can be added to x_0 .

5 Fundamental representations of $SL(n)$

5.1 Intertwining operators

This example is practically identical to example 1, however, we deal with it in a more “algebraic” way to demonstrate some crucial points of the approach.

There are as many as $r \equiv \text{rank } G = n - 1$ fundamental representations of $SL(n)$. Let us begin with the simplest fundamental representation F - the n -plet, consisting of the states

$$\psi_i = T_-^{i-1} |0\rangle, \quad i = 1, \dots, n. \quad (62)$$

Here the distinguished generator T_- is essentially a sum of those for all the r *simple* roots of G : $T_- = \sum_{i=1}^r T_{-\alpha_i}$. Then all the other fundamental representations $F^{(k)}$ are defined as skew powers of $F = F^{(1)}$:

$$F^{(k)} = \left\{ \Psi_{i_1 \dots i_k}^{(k)} \sim \psi_{[i_1} \dots \psi_{i_k]} \right\} \quad (63)$$

$F^{(k)}$ is essentially generated by the operators

$$R_k(T_-^i) \equiv T_-^i \otimes I \otimes \dots \otimes I + I \otimes T_-^i \otimes \dots \otimes I + I \otimes I \otimes \dots \otimes T_-^i. \quad (64)$$

These operators commute with each other. It is clear that for given k exactly k of them (with $i = 1, \dots, k$) are independent.

The intertwining operators which are of interest for us are

$$\begin{aligned} I_{(k)} : F^{(k+1)} &\longrightarrow F^{(k)} \otimes F, & I_{(k)}^* : F^{(k-1)} &\longrightarrow F^* \otimes F^{(k)}, \\ \text{and } \Gamma_{k|k'} : F^{(k+1)} \otimes F^{(k'-1)} &\longrightarrow F^{(k)} \otimes F^{(k')}. \end{aligned} \quad (65)$$

Here

$$\begin{aligned} F^* = F^{(r)} &= \left\{ \psi^i \sim \epsilon^{i i_1 \dots i_r} \psi_{[i_1} \dots \psi_{i_r]} \right\}, \\ I_{(k)} : \Psi_{i_1 \dots i_{k+1}}^{(k+1)} &= \Psi_{[i_1 \dots i_k} \psi_{i_{k+1}]}, & I_{(k)}^* : \Psi_{i_1 \dots i_{k-1}}^{(k-1)} &= \Psi_{i_1 \dots i_{k-1} i}^{(k)} \psi^i, \end{aligned} \quad (66)$$

and $\Gamma_{k|k'}$ is constructed with the help of embedding $I \longrightarrow F \otimes F^*$, induced by the pairing $\psi_i \psi^i$: the basis in linear space $F^{(k+1)} \otimes F^{(k'-1)}$, induced by $\Gamma_{k|k'}$ from that in $F^{(k)} \otimes F^{(k')}$ is:

$$\Psi_{[i_1 \dots i_k}^{(k)} \Psi_{i_{k+1}] i'_1 \dots i'_{k'-1}}^{(k')}. \quad (67)$$

Operation Γ can be now rewritten in terms of matrix elements

$$g^{(k)} \begin{pmatrix} i_1 \dots i_k \\ j_1 \dots j_k \end{pmatrix} \equiv \langle \Psi_{i_1 \dots i_k} | g | \Psi_{j_1 \dots j_k} \rangle = \det_{1 \leq a, b \leq k} g_{j_b}^{i_a} \quad (68)$$

as follows:

$$g^{(k)} \begin{pmatrix} i_1 \dots i_k \\ j_1 \dots j_k \end{pmatrix} g^{(k')} \begin{pmatrix} i'_1 \dots i'_k \\ j_{k+1} j'_1 \dots j'_{k'-1} \end{pmatrix} = g^{(k+1)} \begin{pmatrix} i_1 \dots i_k [i'_{k'}] \\ j_1 \dots j_{k+1} \end{pmatrix} g^{(k'-1)} \begin{pmatrix} i'_1 \dots i'_{k'-1} \\ j'_1 \dots j'_{k'-1} \end{pmatrix} \quad (69)$$

This is the explicit expression for eq.(15) in the case of fundamental representations, and it is certainly identically true for any $g^{(k)}$ of form (68).

5.2 τ -function

Now let us introduce time-variables and rewrite (69) in terms of τ -functions. We shall denote time variables through s_i, \bar{s}_i , $i = 1, \dots, r$ in order to emphasize their difference from generic t_α, \bar{t}_α labeled by all the positive roots α of G . Note that in order to have a closed system of equations we need to introduce all the r times s_i for all $F^{(k)}$ (though $\tau^{(k)}$ actually depends only on k independent combinations of these).

Since the highest weight of representation $F^{(k)}$ is identified as

$$|F^{(k)}\rangle = |\Psi_{1\dots k}^{(k)}\rangle, \quad (70)$$

we have:

$$\tau_k(s, \bar{s} \mid g) = \langle \Psi_{1\dots k}^{(k)} | \exp\left(\sum_i s_i R_k(T_+^i)\right) g \exp\left(\sum_i \bar{s}_i R_k(T_-^i)\right) | \Psi_{1\dots k}^{(k)} \rangle. \quad (71)$$

Now,

$$\exp\left(\sum_i s_i R_k(T^i)\right) = \exp\left(R_k\left(\sum_i s_i T^i\right)\right) = \left(\exp\left(\sum_i s_i T^i\right)\right)^{\otimes k} = \left(\sum_j P_j(s) T^j\right)^{\otimes k}, \quad (72)$$

where we used the definition of the Schur polynomials

$$\exp\left(\sum_i s_i z^i\right) = \sum_j P_j(s) z^j, \quad (73)$$

their essential property being:

$$\partial_{s_i} P_j(s) = (\partial_{s_1})^i P_j(s) = P_{j-i}(s). \quad (74)$$

Because of (72), we can rewrite the r.h.s. of (71) as

$$\begin{aligned}\tau_k(s, \bar{s} \mid g) &= \sum_{\substack{i_1, \dots, i_k \\ j_1, \dots, j_k}} P_{i_1}(s) \dots P_{i_k}(s) \langle \Psi_{1+i_1, 2+i_2, \dots, k+i_k}^{(k)} \mid g \mid \Psi_{1+j_1, 2+j_2, \dots, k+j_k}^{(k)} \rangle P_{j_1}(\bar{s}) \dots P_{j_k}(\bar{s}) = \\ &= \det_{1 \leq \alpha, \beta \leq k} H_{\beta}^{\alpha}(s, \bar{s}),\end{aligned}\tag{75}$$

where

$$H_{\beta}^{\alpha}(s, \bar{s}) = \sum_{i,j} P_{i-\alpha}(s) g_j^i P_{j-\beta}(\bar{s}).\tag{76}$$

This formula can be considered as including infinitely many times s_i and \bar{s}_i , and it is only due to the finiteness of matrix $g_j^i \in SL(n)$ that H -matrix is additionally constrained

$$\left(\frac{\partial}{\partial s_1} \right)^n H_{\beta}^{\alpha} = 0, \quad \dots \quad \frac{\partial}{\partial s_i} H_{\beta}^{\alpha} = 0, \quad \text{for } i \geq n.\tag{77}$$

The characteristic property of H_{β}^{α} is that it satisfies the following “shift” relations (see (74)):

$$\frac{\partial}{\partial s_i} H_{\beta}^{\alpha} = H_{\beta}^{\alpha+i}, \quad \frac{\partial}{\partial \bar{s}_i} H_{\beta}^{\alpha} = H_{\beta+i}^{\alpha}.\tag{78}$$

Expressions (75), (76) and (78) are, of course, familiar from the theory of KP and Toda hierarchies (see [1, 6, 7, 8] and references therein).

5.3 Bilinear identities

BI (69) can be easily rewritten in terms of H -matrix: by convoluting them with Schur polynomials. Let us denote $H \left(\begin{smallmatrix} \alpha_1 \dots \alpha_k \\ \beta_1 \dots \beta_k \end{smallmatrix} \right) = \det_{1 \leq a, b \leq k} H_{\beta_b}^{\alpha_a}$. In

accordance with this notation $\tau_k = H \left(\begin{smallmatrix} 1 \dots k \\ 1 \dots k \end{smallmatrix} \right)$, while the BI turns into:

$$H \left(\begin{smallmatrix} \alpha_1 \dots \alpha_k \\ [\beta_1 \dots \beta_k] \end{smallmatrix} \right) H \left(\begin{smallmatrix} \alpha'_k \alpha'_1 \dots \alpha'_{k-1} \\ \beta_{k+1} \beta'_1 \dots \beta'_{k-1} \end{smallmatrix} \right) = H \left(\begin{smallmatrix} \alpha_1 \dots \alpha_k [\alpha'_k] \\ \beta_1 \dots \beta_k \beta_{k+1} \end{smallmatrix} \right) H \left(\begin{smallmatrix} \alpha'_1 \dots \alpha'_{k-1} \\ \beta'_1 \dots \beta'_{k-1} \end{smallmatrix} \right). \quad (79)$$

Just like original (69) these are merely matrix identities, valid for any H_β^α . However, after the switch from g to H we, first, essentially represented the equations in n -independent form and, second, opened the possibility to rewrite them in terms of time-derivatives.

For example, in the simplest case of

$$\begin{aligned} \alpha_i &= i, \quad i = 1, \dots, k'; & \beta_i &= i, \quad i = 1, \dots, k+1; \\ \alpha'_i &= i, \quad i = 1, \dots, k-1, \quad \alpha'_k = k+1; & \beta'_i &= i, \quad i = 1, \dots, k-1 \end{aligned} \quad (80)$$

we get:

$$\begin{aligned} H \left(\begin{smallmatrix} 1 \dots k \\ 1 \dots k \end{smallmatrix} \right) H \left(\begin{smallmatrix} k+1, 1 \dots k-1 \\ k+1, 1 \dots k-1 \end{smallmatrix} \right) - H \left(\begin{smallmatrix} 1 \dots k-1, k \\ 1 \dots k-1, k+1 \end{smallmatrix} \right) H \left(\begin{smallmatrix} k+1, 1 \dots k-1 \\ k, 1 \dots, k-1 \end{smallmatrix} \right) = \\ = H \left(\begin{smallmatrix} 1 \dots k+1 \\ 1 \dots k+1 \end{smallmatrix} \right) H \left(\begin{smallmatrix} 1 \dots k-1 \\ 1 \dots k-1 \end{smallmatrix} \right) \end{aligned} \quad (81)$$

(all other terms arising in the process of symmetrization vanish). This in turn can be represented through τ -functions:

$$\partial_1 \bar{\partial}_1 \tau_k \cdot \tau_k - \bar{\partial}_1 \tau_k \partial \tau_k = \tau_{k+1} \tau_{k-1}. \quad (82)$$

This is the usual lowest Toda lattice equation. For finite n the set of solutions is labeled by $g \in SL(n)$ as a result of additional constraints (77).

6 τ -functions and Satsuma hierarchy

6.1 τ -functions and representations of algebra of functions

Non-commutative τ -function in our definition is an element of algebra of functions on the group. Therefore, one can fix different representations of this algebra and look at “the values” of the corresponding τ -functions. The natural question arises how one can come to some c -number functions in this way. The simplest c -number object is “double generating function”, which generates matrix elements of both representation and co-representation of the universal enveloping algebra (co-representation of the universal enveloping algebra = representation of the algebra of functions). This double generating function depends on four sets of time variables, and should satisfy BI with respect to both representation and co-representation indices (therefore, this should be some 4-dimensional system of equations).

Another c -number function is the τ -function itself taken in trivial co-representation. In order to understand better all this stuff let us discuss what is the structure of co-representations in our usual example of $SL_q(2)$.

Co-representations are given by the corresponding representations of algebra (60). This is a Borel algebra, and, therefore, it has no finite-dimensional non-trivial irreps [14]. All finite-dimensional representations are reducible, but not completely reducible. Let us write down manifestly the connection (bosonization) of the standard generators of $A(SL_q(2))$ (47) in terms of algebra (60):

$$a = e^{\frac{1}{2}x_0} + x_+x_-e^{-\frac{1}{2}x_0}, \quad b = x_+e^{-\frac{1}{2}x_0}, \quad c = e^{-\frac{1}{2}x_0}x_-, \quad d = e^{-\frac{1}{2}x_0}. \quad (83)$$

This expression coincides with (6), the commutation relations between x 's being only different.

Thus, there are only two irreps: the trivial one, which is given by $x_+ = x_- = 0$, i.e. $ad = 1$, $b = c = 0$, and the infinite dimensional irrep, given manifestly by the action on the basis $\{e_k\}_{k \geq 0}$ [15]

$$ae_k = (1 - q^{2k})^{\frac{1}{2}} e_{k-1} \quad (ae_0 = 0), \quad de_k = (1 - q^{2k+2})^{\frac{1}{2}} e_{k+1}, \quad ce_k = \theta q^k e_k, \quad be_k = -\theta^{-1} q^{k+1} e_k. \quad (84)$$

This picture can be easily generalized to other quantum groups (of the rank r), as the corresponding algebras of x 's are always Borel algebras. Therefore, there are only trivial and infinite-dimensional (both r -parametric) irreps of the algebras of functions in these cases [15].

τ -function in trivial representation

It was already mentioned that, taken in trivial representation of the algebra of functions, τ -function presents another example of c -number function. Hence, this particular example is of great importance. Let us first consider the simplest case of $SL_q(2)$. Then, there is no analog of the determinant expression (75) for the τ -function. Indeed, let us introduce (hereafter we denote $D \equiv D^{(0)}$):

$$H_1^1 = \tau_F = a + b\bar{t} + ct + dt\bar{t}. \quad (85)$$

If

$$H_2^1 = D_{\bar{t}} H_1^1 = b + dt, \quad H_1^2 = D_t H_1^1 = c + d\bar{t}, \quad H_2^2 = D_{\bar{t}} D_t H_1^1 = d, \quad (86)$$

we see that H_b^a is actually not lying in $SL_q(2)$ (for example, $H_2^1 H_1^2 \neq H_1^2 H_2^1$), i.e. the matrix consisting of the τ_F and its derivatives, despite these are all

elements of $A(G)$, does not longer belong to G_q . Thus, it is not reasonable to consider $\det_q H$ (or the definition of H should be somehow modified). Instead the appropriate formula for the case of $SL_q(2)$ looks like

$$\tau_{F(2)} = \det_q g = 1 = H_1^1 H_2^2 - q H_2^1 M_t^- H_1^2 = \tau_F D_t D_{\bar{t}} \tau_F - q D_{\bar{t}} \tau_F M_t^- D_t \tau_F. \quad (87)$$

Now one can take trivial representation of $A(SL_q(2))$: $ad = 1$, $b = c = 0$ and obtain that the equation (this was first proposed by Satsuma et al. [16])

$$\tau_F D_t D_{\bar{t}} \tau_F - D_{\bar{t}} \tau_F D_t \tau_F = 1 \quad (88)$$

is correct. This equation looks like (82) and has the form independent of the concrete $SL_q(2)$ algebra. Moreover, equation (87) seems not to be generalizable to $SL_q(n)$ -case, in contrast to (88) (being applied only to τ -functions in trivial representations).

Another argument in favor of equation (88) is that it immediately leads to determinants of q -Schur polynomials. On the other hand, the appearance of these is rather natural in the trivial representations. Indeed, let us consider unit element g (which corresponds to taking trivial representation of $A(G)$). Then, for $SL_q(n)$ algebra, one obtains by direct calculation [2]

$$\langle k, 0, \dots, 0 | e_{q^{-1}}^{s_2 T_{12}} e_{q^{-1}}^{s_3 T_{13}} \dots e_{q^{-1}}^{s_n T_{1n}} \times e_q^{\bar{s}_2 T_{21}} e_q^{\bar{s}_3 T_{31}} \dots e_q^{\bar{s}_n T_{n1}} | k, 0, \dots, 0 \rangle = P_k^{(q)}(s\bar{s}), \quad (89)$$

where $\langle k, 0, \dots, 0 |$ denotes the symmetric product of k simplest fundamental representations.

Because of all these reasons we are led to consider equation (88) in more details.

6.2 Satsuma difference hierarchy

Satsuma hierarchy from Toda lattice hierarchy

Now we demonstrate that the *difference* (Satsuma) equation (88) can be obtained in the framework of the standard *differential* Toda lattice ($GL(\infty)$) hierarchy by the redefinition of time flows [3]. Indeed, Eq.(88) is a corollary of *two* statements: the basic identity (69) and the particular definition (1), which in this case implies (76) with P 's being ordinary Schur polynomials (73). At least, in this simple situation (of fundamental representations of $SL(n)$) one could define τ -function not by eq.(1), but just by eq.(75), with

$$H_\beta^\alpha(s, \bar{s}) \longrightarrow \mathcal{H}_\beta^\alpha(s, \bar{s}) = \sum_{i,j} \mathcal{P}_{i-\alpha}(s) g_j^i \mathcal{P}_{j-\beta}(\bar{s}) \quad (90)$$

with *any* set of independent functions (not even polynomials) \mathcal{P}_α . Such

$$\tau_k^{(\mathcal{P})} = \det_{1 \leq \alpha, \beta \leq k} \mathcal{H}_\beta^\alpha \quad (91)$$

still remains a generating function for all matrix elements of $G = SL(n)$ in representation $F^{(k)}$. This freedom should be kept in mind when dealing with “generalized τ -functions”. As a simple example, one can take $\mathcal{P}_\alpha(s)$ to be q -Schur polynomials,

$$\prod_i e_{q^i}(s_i z^i) = \sum_j P_j^{(q)}(s) z^j, \quad (92)$$

which satisfy

$$D_{s_i} P_j^{(q)}(s) = (D_{s_1})^i P_j^{(q)}(s) = P_{j-i}^{(q)}(s). \quad (93)$$

Then instead of (78) we would have:

$$D_{s_i} \mathcal{H}_\beta^\alpha = \mathcal{H}_{\beta}^{\alpha+i}, \quad D_{\bar{s}_i} \mathcal{H}_\beta^\alpha = \mathcal{H}_{\beta+i}^\alpha \quad (94)$$

and

$$\tau_k^{(P^{(q)})}(s, \bar{s}|g) = \det_{1 \leq \alpha, \beta \leq k} D_{s_1}^{\alpha-1} D_{\bar{s}_1}^{\beta-1} \mathcal{H}_1^1(s, \bar{s}). \quad (95)$$

So defined τ -function satisfies (88) [16, 3]:

$$\tau_k \cdot D_{s_1} D_{\bar{s}_1} \tau_k - D_{s_1} \tau_k \cdot D_{\bar{s}_1} \tau_k = \tau_{k-1} \cdot M_{s_1}^+ M_{\bar{s}_1}^+ \tau_{k+1}, \quad (96)$$

...

where ... means other equations of the (Satsuma) hierarchy. The simplest way to prove this formula is to rewrite τ -function using

$$\det D_{s_1}^i D_{\bar{s}_1}^j H = q^{-(N-1)(N-2)} (1-q)^{N(N-1)} (tt)^{\frac{N(N-1)}{2}} \det_{0 \leq i, j < N} (M_{s_1}^+)^i (M_{\bar{s}_1}^+)^j H \quad (97)$$

and then to apply to this τ -function the Jacobi identity. The Jacobi identity is a particular ($p = 2$) case of the general identity for the minors of any matrix,

$$\sum_{i_p} H_{ri_p} \hat{H}_{i_1 \dots i_p | j_1 \dots j_p} = \frac{1}{p!} \sum_P (-)^P \hat{H}_{i_1 \dots i_{p-1} | j_{P(1)} \dots j_{P(p-1)}} \delta_{rj_{P(p)}}, \quad (98)$$

where the sum on the r.h.s. is over all permutations of the p indices and $\hat{H}_{i_1 \dots i_p | j_1 \dots j_p}$ denotes the determinant (minor) of the matrix, which is obtained from H_{ij} by removing the rows $i_1 \dots i_p$ and the columns $j_1 \dots j_p$. Using the fact that $(H^{-1})_{ij} = \hat{H}_{i|j} / \hat{H}$, this identity can be rewritten as

$$\hat{H} \hat{H}_{i_1 \dots i_p | j_1 \dots j_p} = \left(\frac{1}{p!} \right)^2 \sum_{P, P'} (-)^P (-)^{P'} \hat{H}_{i_{P'(1)} \dots i_{P'(p-1)} | j_{P(1)} \dots j_{P(p-1)}} \delta_{i_{P'(p)} | j_{P(p)}}. \quad (99)$$

Fermionic approach to Satsuma hierarchy

Let us look at the fermionic description of the Satsuma hierarchy. As far as this hierarchy is obtained from the Toda lattice hierarchy (see example 1)

by the redefiniton of time variables, one can immediately use the τ -function of the Toda lattice hierarchy, and just substitute new times. Indeed,

$$\prod_{k=1}^{\infty} e_{q^k}(s_k z^k) = \prod_{k=1}^{\infty} e^{t_k z^k}, \quad (100)$$

provided the t 's are expressed in terms of the s 's according to

$$\sum_{k=1}^{\infty} t_k z^k = \sum_{n,k=1}^{\infty} \frac{s_k^n (1 - q_k)^n}{n(1 - q_k^n)} z^{nk}. \quad (101)$$

Thus

$$P_k^{(q)}(s) = P_k(t). \quad (102)$$

Because of this, τ -function can be represented as

$$\tau_n(s, \bar{s}|g) = \tau_n(t, \bar{t}|g) \stackrel{(2)}{=} \langle F^{(n)} | e^{H\{t\}} g e^{\bar{H}\{\bar{t}\}} | F^{(n)} \rangle \quad (103)$$

and

$$\begin{aligned} H\{t\} &= \sum_{n>0} t_n J_{+n} \stackrel{(101)}{=} \sum_{n,k=0}^{\infty} \frac{s_k^n (1 - q_k)^n}{n(1 - q_k^n)} J_{+nk}, \\ \bar{H}\{\bar{t}\} &= \sum_{n>0} \bar{t}_n J_{-n} = \sum_{n,k=0}^{\infty} \frac{\bar{s}_k^n (1 - q_k)^n}{n(1 - q_k^n)} J_{-nk}. \end{aligned} \quad (104)$$

The Satsuma τ -function can be also considered as some Miwa transformed Toda τ -function [3]. Indeed, the general Miwa transformation of times would be $t_n = \frac{1}{n} \sum_i \lambda^{-i}$ with sum running over generally infinite set of integer numbers. Then, using formulas analogous to

$$t_k = \frac{1}{k} \frac{((1 - q)s_1)^k}{1 - q^k} = \frac{1}{k} \sum_{l \geq 0} ((1 - q)q^l s_1)^k, \quad (105)$$

one gets the following set of Miwa variables leading to the Satsuma hierarchy

$$\left\{ e^{2\pi i a/k} \lambda_k q_k^{-l/k} \mid a = 0, \dots, k-1; \quad l \geq 0 \right\}, \quad \lambda_k = ((1 - q_k)s_k)^{-1/k}. \quad (106)$$

Thus, after Miwa transformation of the Toda lattice hierarchy, with the specific set of Miwa variables (106) one gets the Satsuma hierarchy.

7 Concluding remarks

In this paper we have introduced the notion of generalized τ -function and demonstrated that it satisfied a set of BI. We also have discussed non-commutative τ -functions arising in the framework of quantum groups.

The generalized τ -function, with associated BI, is supposed to play a very important role in applications. First, it gives a tool to investigate WZW theories with level greater than 1 in integrable treatment. Further development in the same direction might be dealing with 2-loop, 3-loop etc. algebras. Another application of the approach advocated in the present paper is the case of quantum deformed algebras. This case has to have much to do with careful quantization of the Liouville theory [17], and also might shed a light on the famous fact observed in quantum integrable systems, whose correlators (more precisely, their generating functional) satisfy some classical integrable equations [18]. This phenomenon is similar to that observed in the connection of theories of $2d$ gravity and matrix models [8].

Acknowledgements

I am indebted to my collaborators A.Gerasimov, S.Kharchev, S.Khoroshkin, D.Lebedev, A.Morozov and L.Vinet. I also acknowledge A.Zabrodin for numerous discussions. I would like to thank the organizers of the Workshop on Symmetries and Integrability of Difference Equations for the invitation and kind hospitality. The work is partially supported by grant 93-02-03379 of the

Russian Foundation of Fundamental Research and by grant MGK000 of the International Soros Foundation.

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